Trigonometry Without Sines and Geometry Without Angles

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A Discussion and Critique of N. J. Wildberger's *Divine Proportions*

During the fall semester of 2005, one of my students, Wil Wade, told me of a book that he had discovered when doing some online searching. The advertisement told of a radical new approach to geometry that the book propounded. Wil wondered whether I was aware of the book, and I told him that I was not. After checking out the online messages about the book, I thought that I ought to get one and see whether the author was really on to something. I am always looking for better teaching approaches to various mathematical topics. I thought that I should investigate just in case I might want to change the way I teach geometry.

In the author's introduction, he claims: "This book revolutionizes trigonometry, re-evaluates and expands Euclidean geometry, and gives a simpler and more natural approach to many practical geometric problems. This new theory unites the three core areas of mathematics—geometry, number theory and algebra—and expels analysis and infinite processes from the foundations of the subject." The author claims that his approach is an easier way to teach and learn geometry.

Wildberger identifies three "rocks" (his term) on which most attempts at developing Euclidean geometry founder:

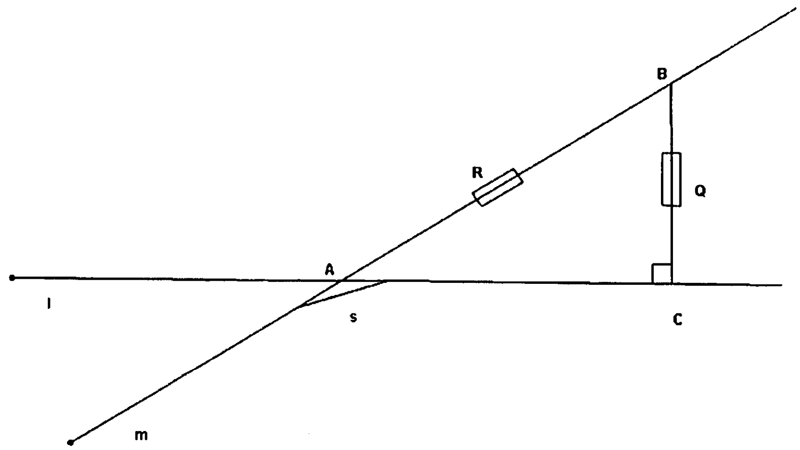
1. the ambiguity of defining the continuum or real number line,
2. the problem of stating precisely what an angle is, and
3. the difficulty of making the jump from two to three dimensions.

The author seems almost to have an Intuitionist or constructivist dislike for the real number system. He attempts to avoid the deeper philosophical and logical problems with this concept by using only rational numbers or square roots of rational numbers throughout his text. I am not sure that he completely bypasses all the problems with the real number continuum. It seems more likely to me that he has simply changed the locus of the difficulties. As for problems with defining angles, I don't believe that the difficulty is in being precise so much as the lack of consistency from author to author in their definitions of angle. Whether going from two to three dimensions poses grave difficulties, I leave for those better versed in geometry than I to ponder.

Two Foundational Concepts

Wildberger's main method of avoiding the irrational and the "ambiguous" is through his concepts of *quadrance* and *spread.* Given points $A_1(x_1,y_1)$ and $A_2(x_2,y_2)$ in the plane, the ***quadrance*** between the two points is given by $Q(A_1;A_2)=(x_1-x_2)^2+(y_1-y_2)^2$. We can see that quadrance is simply the square of the distance between two points. The author wants not to assume that distance has been defined so that he can develop his geometry based solely on the notion of quadrance and thus avoid radicals and irrational distances.

The *spread* is defined in such a way as to make it equivalent to the square of the sine of an angle. Of course, there is no such concept as *angle* in this development of geometry. The spread is thus based on the two lines, as demonstrated in the diagram shown on the following page. Notice the two small rectangular boxes which are labeled Q and R. This is Wildberger's notation for quadrance.



The *spread* of lines l and m is $s(l,m) = Q/R$

Actually, the above relationship, which shows how the spread relates to the usual sine function, is not the way Wildberger defines spread. The square-of-the-sine relationship is actually a theorem in the book. Spread is defined analytically in this manner: Suppose that the two lines l and m have equations $ax + by = c$ and $dx + ey = f$, respectively. Let $s(l,m)$ represent the spread of l and m. Then $s(l,m) = \dfrac{(ae - db)^2}{(a^2 + b^2)(d^2 + e^2)}$.

Triangle Notation

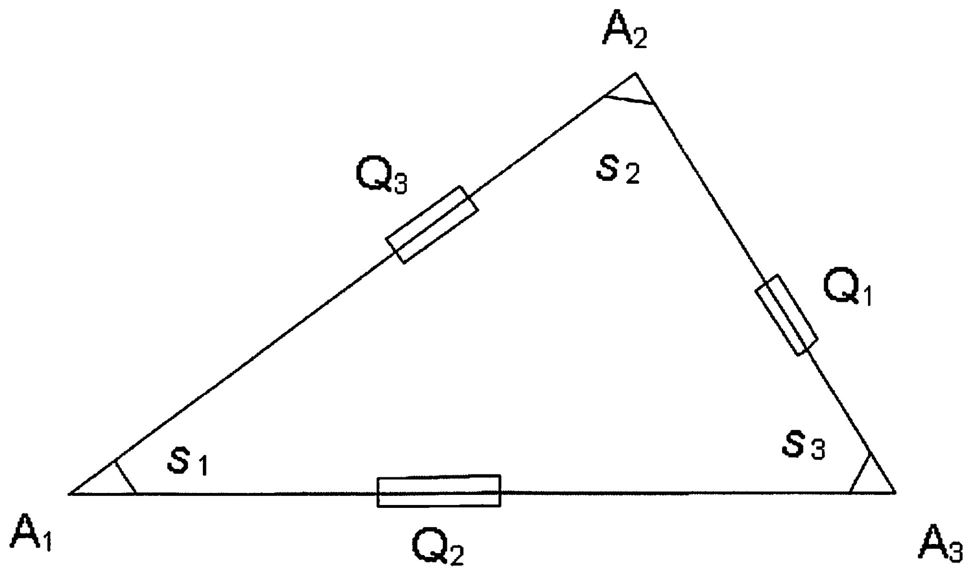
Let A_1, A_2, and A_3 be three points. Define

$Q_1 = Q(A_2, A_3)$

$Q_2 = Q(A_1, A_3)$

$Q_3 = Q(A_1, A_2)$

Let s_1 be the spread of the two lines, determined by the three points above, that intersect at A_1. Similarly, let s_2 and s_3 be the spreads of the lines intersecting at A_2 and A_3, respectively.



From the diagram above, we can see that Wildberger is closely following the usual labeling conventions for triangles. We will assume this labeling in stating the theorems to follow.

The Five Main Laws of Rational Trigonometry

Wildberger identifies five results as foundational for his treatment of geometry. The first is the **Triple Quad Formula**: Three points A_1, A_2, and A_3 are collinear precisely when $(Q_1 + Q_2 + Q_3)^2 = 2(Q_1^2 + Q_2^2 + Q_3^2)$. There is no immediate analog to this theorem in the usual approaches to geometry. Notice that the formula is quadratic in the quadrances; therefore it is a fourth-degree relationship among the actual lengths.

Pythagoras' Theorem: The lines A_1A_3 and A_2A_3 are perpendicular precisely when $Q_1 + Q_2 = Q_3$. This is immediately clear from the definition of the quadrance as the square of the distance. Here is one case where there is certainly a simplification vis-à-vis the traditional theorem.

The analog to the triple quad formula for quadrances is the **Triple Spread Formula:** For any triangle, $(s_1 + s_2 + s_3)^2 = 2(s_1^2 + s_2^2 + s_3^2) + 4\, s_1s_2s_3$. There is no standard theorem of trigonometry to which this relationship alludes. Once again, since the spreads are squares of sines, this formula is actually a fourth-degree relationship among the sines of a triangle.

The final two of the five main theorems will be very obviously linked to standard results in trigonometry. The **Spread Law** states that for any triangle with nonzero quadrances,

$$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3}$$

This is very clearly just the squared version of the traditional Law of Sines. Wildberger has to state that the triangle has nonzero quadrances because he is constructing his geometry over an arbitrary field. Over such a field, it may be possible for the sum of two squares to equal zero.

For any two lines with spread s, define the *cross* of the lines to be simply $1 - s$. Adapting to the triangle notation in the obvious way, the **Cross Law** states $(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2c_3$. We see that the cross is just the square of the cosine, and that the cross law is the rational version of the usual Law of Cosines.

Quadrea

Definition: For three points A_1, A_2, and A_3 with related quadrances Q_1, Q_2, and Q_3 (following the usual notational convention), the ***quadrea*** of the set $\{A_1, A_2, A_3\}$ is the number

$\mathcal{A} = \mathcal{A}(Q_1, Q_2, Q_3) = (Q_1 + Q_2 + Q_3)^2 - 2(Q_1^2 + Q_2^2 + Q_3^2)$

Quadrea is the analog of *area* in rational geometry. Wildberger develops several theorems related to this concept.

Right Quadrea: If a right triangle $\Delta A_1A_2A_3$ has quadrances Q_1, Q_2, and Q_3, with right vertex at A_3, then it has quadrea $\mathcal{A} = 4Q_1Q_2$.

Quadrea Spread: Suppose three points A_1, A_2, and A_3 form quadrances Q_1, Q_2, and Q_3 as usual and spread $s_3 = s(A_3A_1, A_3A_2)$. Then the quadrea of $\{A_1, A_2, A_3\}$ is $\mathcal{A} = 4Q_1Q_2s_3$. This very important theorem forges a link between the quadrances of a triangle and the spreads of its lines.

Archimedes' Formula: Suppose the triangle $\Delta A_1A_2A_3$ has square quadrances $Q_1 = d_1^2$, $Q_2 = d_2^2$, and $Q_3 = d_3^2$ for some numbers d_1, d_2, and d_3. Then the quadrea is given by

$$\mathcal{A} = (d_1+d_2+d_3)\,(d_1+d_2-d_3)\,(d_2+d_3-d_1)\,(d_3+d_1-d_2)$$

You might recognize this as Heron's (Hero's) formula for the area of a triangle (squared and multiplied by 16).

__**An Example Applying the Above Concepts (Example 6.1, p. 83)**__

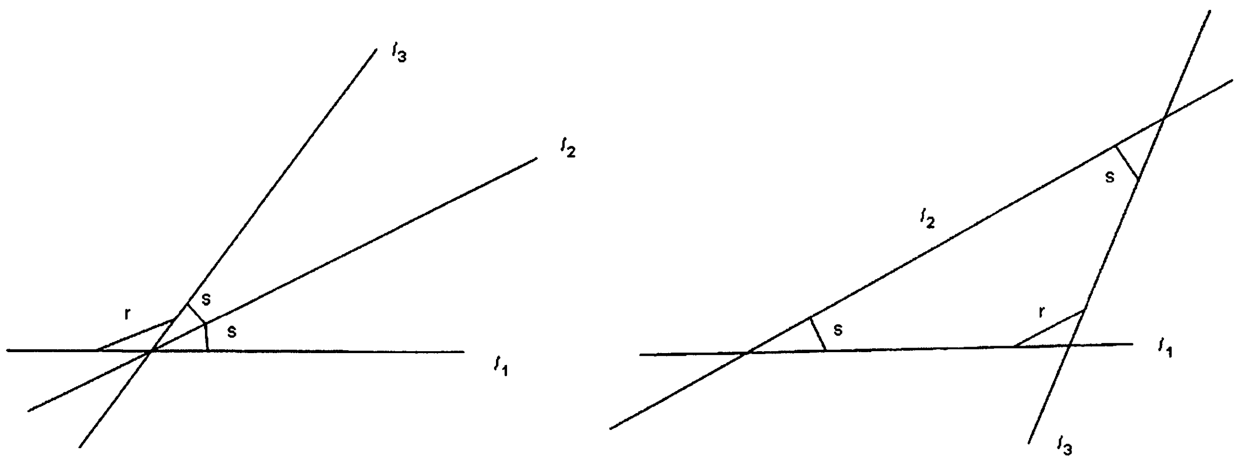
The triangle $\Delta A_1A_2A_3$ with quadrances $Q_1 = 16$, $Q_2 = 36$, and $Q_3 = 9$ has quadrea

$$\mathcal{A} = \mathcal{A}(16,36,9) = (16+36+9)^2 - 2(16^2+36^2+9^2) = 455.$$

So by the quadrea spread theorem, the three spreads of the triangle are given by

$$s_1 = \frac{\mathcal{A}}{4Q_2Q_3} = \frac{455}{4\times 36\times 9} = \frac{455}{1296}$$
$$s_2 = \frac{\mathcal{A}}{4Q_1Q_3} = \frac{455}{4\times 16\times 9} = \frac{455}{576}$$
$$s_3 = \frac{\mathcal{A}}{4Q_1Q_2} = \frac{455}{4\times 16\times 36} = \frac{455}{2304}.$$

One More Illustrative Theorem



Two Possible Relationships Illustrating the Equal Spreads Theorem

Equal Spreads: If l_1, l_2, and l_3 are lines with $s(l_1, l_2) = s(l_2, l_3) = s$ as in either of the diagrams above, then $s(l_1, l_3) = 0$ or $s(l_1, l_3) = 4s(1 - s)$. Notice that this is related to doubling an angle. In the classical development, the angle corresponding to the spread s above and measuring x would be doubled to $2x$ for the angle between l_1 and l_3. Notice also that in the rational trigonometry development, there is no need to distinguish between the angle determined by l_1 and l_3, which has l_2 in its interior, and the supplement of that angle (labeled in the left diagram above with spread r).

Characteristics and Contents

The primary approach to geometry is through algebra, making the approach very strongly analytic rather than synthetic. Concepts are defined in terms of coordinates and parameters of lines. Classical geometrical concepts and relationships are then derived from these definitions as algebraic theorems. Algebra is primary, and geometry is secondary in this approach.

The book is a veritable cornucopia of polynomial identities, most of which are quadratic or related to quadratics. In addition to the theorems mentioned earlier, we have the following representative examples:

Fibonacci's identity: $(x_1y_2 - x_2y_1)^2(x_1x_2 + y_1y_2)^2 = (x_1^2 + x_2^2)(y_1^2 + y_2^2)$

The identity used to prove that the analytical definition of spread is equivalent to the geometric ratio:

$$(y_1 - y_2)(x_3 - x_1) - (y_1 - y_3)(x_2 - x_1) = (y_1 - y_3)(x_3 - x_2) - (y_2 - y_3)(x_3 - x_1)$$

A typical exercise from the book (Exercise 5.7; the first line is a definition): Show that

$$\begin{aligned}
A(a,b,c) &= (a+b+c)^2 - 2(a^2 + b^2 + c^2) \\
&= 4ab - (a+b-c)^2 \\
&= 2(ab+bc+ca) - (a^2 + b^2 + c^2) \\
&= 4(ab+bc+ca) - (a+b+c)^2 \\
&= \begin{vmatrix} 2a & a+b-c \\ a+b-c & 2b \end{vmatrix} \\
&= -\begin{vmatrix} 0 & a & b & 1 \\ a & 0 & c & 1 \\ b & c & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}.
\end{aligned}$$

Brahmagupta's identity: Suppose that $Q_{12} = d_{12}^2, Q_{23} = d_{23}^2, Q_{34} = d_{34}^2$, and $Q_{14} = d_{14}^2$ for some numbers d_{12}, d_{23}, d_{34}, and d_{14}. Then

$$\left((Q_{12} + Q_{23} + Q_{34} + Q_{14})^2 - 2(Q_{12}^2 + Q_{23}^2 + Q_{34}^2 + Q_{14}^2)\right)^2 - 64Q_{12}Q_{23}Q_{34}Q_{14} =$$

$$\begin{aligned}
&(d_{12} - d_{14} + d_{23} + d_{34})(d_{12} + d_{14} + d_{23} - d_{34})(d_{14} - d_{12} + d_{23} + d_{34})(d_{12} + d_{14} - d_{23} + d_{34}) \\
&\times (d_{12} + d_{14} + d_{23} + d_{34})(d_{12} - d_{14} - d_{23} + d_{34})(d_{12} - d_{14} + d_{23} - d_{34})(d_{23} - d_{14} - d_{12} + d_{34}).
\end{aligned}$$

(The purpose of the double subscripts above is to relate the values in the identity to the quadrances of a quadrilateral, with the triangle notation extended to four-sided objects in the obvious way.)

Topics in the book cover a wide range of the usual elementary geometry subjects (triangles, isometries, regular polygons, conics, circles, quadrilaterals, etc.). The book is rich with applications to problems related to the topics above and also three-dimensional geometry, physics, surveying, Platonic solids, and rational spherical coordinates. The author promises a future volume to include non-Euclidean geometries, which can be merged with Euclidean geometry to form something he calls *chromo-geometry.*

The author's aim is to avoid some of the logical and pedagogical difficulties encountered in standard treatments due to reliance upon properties of the real number system. To this end, he tries to create a geometry that is valid over *any* field: rational, real, complex, or even finite. Certain strange things happen when the underlying field is finite or has non-zero characteristic. A *null line* is one of the form $ax + by = c$ where $a^2 + b^2 = 0$. Such lines cause problems in geometry and often must be excluded from consideration. For example, one cannot use the formula to find the quadrance from a point to a line if the line is null. It is also true that two distinct points may determine a quadrance of zero. These exceptions must always be kept in mind if one attempts to develop his theory in such a way as to take all possible fields into account.

<u>Strengths of this Approach to Geometry</u>

- Deep familiarity with the properties of the real number system is not a prerequisite.
- The ambiguity of the standard definition of *angle* is avoided.
- All definitions are clear-cut and more intuitive than the classical treatment.

- No transcendental functions or relationships are required.
- Most problems are accessible—at least in theory—to hand computation and in most cases avoid the use of radicals.
- Geometry is given a solid, rigorous basis that is anchored in the properties of its underlying field.
- The geometry and trigonometry thus developed can be used successfully to solve problems in every situation where the classical subjects are applied.
- The author does succeed in his attempt at a unified theory that is not dependent upon the properties of a particular field.

Weaknesses of this Approach

- Much that is intuitive to the classical approach is given up here.
- We no longer have additivity of either lengths or angles.
- Geometry over finite fields is almost purely algebraic and non-intuitive.
- I found the inclusion of the finite fields to be an annoying distraction to the development.
- The heavy algebraic/analytic approach contributes to the non-intuitive nature of the development.
- A student would have to be very, very good in algebra to be comfortable with the arguments justifying the theorems.

- Many of the problems that theoretically could be solved by hand are in reality very complicated, even though they may involve only rational quantities.
- For these reasons I do not feel that we should abandon our customary approach to teaching geometry in favor of this new method.

Addendum: An Example of Solving a Triangle—Example 10.1, pp. 129-130

If two quadrances and a spread, say s_3, are known, then the cross law $(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2(1 - s_3)$ gives a quadratic equation for the third unknown quadrance.

Example 10.1: In the rational number field $\Delta A_1A_2A_3$ has $s_3 = 81/130$, $Q_1 = 5$, and $Q_3 = 17$. Now use the cross law to get

$$(Q_2 - 12)^2 = 4 \times 5 \times Q_2 \times 49/130 = 98/13 \times Q_2.$$

Rearrange to find $(13\, Q_2 - 72)(Q_2 - 26) = 0$.

Thus i) $Q_2 = 72/13$ or ii) $Q_2 = 26$.

i) If $Q_2 = 72/13$, the spread law gives $\dfrac{s_1}{5} = \dfrac{s_2}{72/13} = \dfrac{81/130}{17}$; so that $s_1 = 81/442$, $s_2 = 2916/14{,}365$, and $s_3 = 81/130$.

ii) If $Q_2 = 26$, the spread law gives $\dfrac{s_1}{5} = \dfrac{s_2}{26} = \dfrac{81/130}{17}$; in which case $s_1 = 81/442$, $s_2 = 81/85$, and $s_3 = 81/130$.

These two possibilities are illustrated to scale in the figure below, which is as it appears in the book itself.

130 10. TRIAN

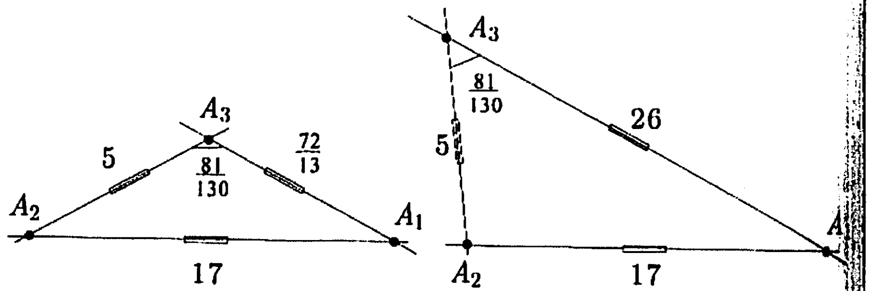


Figure 10.8: Two possibilities

Bibliographical Information

N. J. Wildberger. *Divine Proportions: Rational Trigonometry to Universal Geometry.* Wild Egg Pty Ltd., 2005.

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webpages:

- http://web.maths.unsw.edu.au/~norman/
- http://wildegg.com

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